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Abstract. We prove approximate Lipschitz stability for non-overdetermined inverse scattering at fixed energy with incomplete data in dimension $d \geq 2$. Our estimates are given in uniform norm for coefficient difference and related stability precision efficiently increases with increasing energy and coefficient difference regularity. In addition, our estimates are rather optimal even in the Born approximation.

1. Introduction

We consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \ d \geq 2, \ E > 0, \tag{1.1}$$

where

$$v \ \text{ is real-valued}, \quad v \in L^\infty_\sigma(\mathbb{R}^d) \ \text{ for some } \ \sigma > d, \tag{1.2}$$

where

$$\begin{aligned} &L^\infty_\sigma(\mathbb{R}^d) = \{u \in L^\infty(\mathbb{R}^d) : \ \|u\|_\sigma < +\infty\}, \\ &\|u\|_\sigma = ess \sup_{x\in\mathbb{R}^d} (1+|x|)^\sigma |u(x)|, \sigma \geq 0. \end{aligned} \tag{1.3}$$

For equation (1.1) we consider the scattering amplitude f on $\mathcal{M}_E$,

$$\mathcal{M}_E = \{k \in \mathbb{R}^d, \ l \in \mathbb{R}^d : \ k^2 = l^2 = E\}, \ E > 0. \tag{1.4}$$

For definitions of the scattering amplitude, see formula (1.5) below and, for example, reviews given in [F2], [FM]. The scattering amplitude f arises, in particular, as a coefficient with scattered spherical wave $e^{i|k||x|}/|x|^{(d-1)/2}$ in the asymptotics of the wave solutions $\psi^+(x,k)$ describing scattering of incident plan wave e^{ikx} for equation (1.1):

$$\psi^+(x,k) = e^{ikx} + c(d,|k|)\frac{e^{i|k||x|}}{|x|^{(d-1)/2}} f(k, |k|\frac{x}{|x|}) + o\big(\frac{1}{|x|^{(d-1)/2}}\big) \ \text{ as } \ |x| \to \infty, \tag{1.5}$$

where $x \in \mathbb{R}^d, \ k \in \mathbb{R}^d, \ k^2 = E, \ c(d,|k|) = -\pi i(-2\pi i)^{(d-1)/2}|k|^{(d-3)/2}$.

Given v, to determine f one can use, in particular, the Lippmann-Schwinger integral equation

$$\begin{aligned} &\psi^+(x,k) = e^{ikx} + \int_{\mathbb{R}^d} G^+(x-y,k)v(y)\psi^+(y,k)dy, \\ &G^+(x,k) = -\left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \frac{e^{i\xi x}d\xi}{\xi^2 - k^2 - i0}, \end{aligned} \tag{1.6}$$

and the formula

$$f(k,l) = \left(\frac{1}{2\pi}\right)^d \int\limits_{\mathbb{R}^d} e^{-ily} v(y)\psi^+(y,k)dy, \tag{1.7}$$

where $x, k, l \in \mathbb{R}^d$, $k^2 = l^2 = E$; see, for example, [BS], [F2].

In the present work, in addition to f on $\mathcal{M}_E$, we consider $f\big|_{\Gamma_E}$ and $f\big|_{\Gamma_E^\tau}$, where

$$\begin{aligned} \Gamma_E &= \{k = \frac{p}{2} + \eta_E(p), l = -\frac{p}{2} + \eta_E(p) :\ p \in \mathcal{B}_{2\sqrt{E}}\}, \\ \Gamma_E^\tau &= \{k = \frac{p}{2} + \eta_E(p), l = -\frac{p}{2} + \eta_E(p) :\ p \in \mathcal{B}_{2\tau\sqrt{E}}\}, \end{aligned} \tag{1.8}$$

$$\mathcal{B}_r = \{p \in \mathbb{R}^d :\ |p| \le r\}, \quad r > 0, \tag{1.9}$$

where $E > 0$, $\tau \in]0,1]$, η_E is a piecewise continuous vector-function of $p \in \mathcal{B}_{2\sqrt{E}}$ such that

$$\eta_E(p)p = 0, \quad \frac{p^2}{4} + (\eta_E(p))^2 = E, \quad p \in \mathcal{B}_{2\sqrt{E}}. \tag{1.10}$$

Note that

$$\begin{aligned} &\Gamma_E^\tau \subseteq \Gamma_E \subset \mathcal{M}_E, \\ &dim\, \mathcal{M}_E = 2d-2, \quad dim\, \Gamma_E^\tau = dim\, \Gamma_E = d, \end{aligned} \tag{1.11}$$

where $E > 0$, $\tau \in]0,1]$, $d \ge 2$.

We consider the following inverse scattering problems for equation (1.1) under assumptions (1.2):

Problem 1.1. Given f on $\mathcal{M}_E$ at fixed $E > 0$, find v on $\mathbb{R}^d$ (at least approximately).

Problem 1.2. Given f on Γ_E^τ at fixed $E > 0$, $\tau \in]0,1]$, find v on $\mathbb{R}^d$ (at least approximately).

Using (1.11) one can see that Problem 1.1 is overdetermined for $d \ge 3$, whereas Problem 1.2 is non-overdetermined.

There are many important results on Problem 1.1, see [ABR], [B], [BAR], [E], [ER2], [F1], [G], [HH], [HN], [I], [IN2], [N1]-[N5], [S1], [VW], [W], [WY] and references therein. On the other hand, to our knowledge, Problem 1.2 was not yet considered explicitly in the literature. Concerning known results for some other non-overdetermined multi-dimensional coefficient inverse problems, see [BK], [ER1], [HN], [K], [N6], [S2] and references therein.

Problems 1.1, 1.2 can be also considered as examples of ill-posed problems; see [BK], [LRS] for an introduction to this theory.

In the present work we obtain approximate Lipschitz stability estimates for Problem 1.2 (with $\tau = \tau(E) = \varepsilon\, E^{(1-d)/(2d)}$ for $E \ge 1$) in dimension $d \ge 2$, see Theorem 2.1 of Section 2. Our estimates are given in uniform norm for coefficient difference and related stability precision efficiently increases with increasing energy and coefficient difference regularity. In addition, at the end of Section 2, we show that our estimates of Theorem 2.1 are rather optimal even for the case of the Born approximation (that is in the linear approximation near zero potential). Our new estimates are much different but coherent with respect to results of [N4], [N5] for Problem 1.1.

2. Stability estimates

Let

$$\begin{aligned} &W^{n,1}(\mathbb{R}^d) = \{u:\ \partial^J u \in L^1(\mathbb{R}^d),\ |J| \le n\},\\ &\|u\|_{n,1} = \max_{|J|\le n} \|\partial^J u\|_{L^1(\mathbb{R}^d)}, \end{aligned} \tag{2.1}$$

where

$$J \in (\mathbb{N}\cup 0)^d,\ |J| = \sum_{i=1}^{d} J_i,\ \partial^J u(x) = \frac{\partial^{|J|}u(x)}{\partial x_1^{J_1}\dots\partial x_d^{J_d}},\ n \in (\mathbb{N}\cup 0).$$

Let $C(\mathcal{M}_E)$ denote continuous functions on $\mathcal{M}_E$ and $C(\Gamma_E)$, $C(\Gamma_E^\tau)$ denote the restrictions of $C(\mathcal{M}_E)$ on Γ_E and Γ_E^τ. Let

$$\begin{aligned} &\|f\|_{C(\Gamma_E^\tau)} = \|f\|_{C(\Gamma_E^\tau),0},\\ &\|f\|_{C(\Gamma_E^\tau),\sigma} = \sup_{(k,l)\in\Gamma_E^\tau} (1+|k-l|)^\sigma |f(k,l)|, \end{aligned} \tag{2.2}$$

where $E > 0$, $0 < \tau \le 1$, $\sigma \ge 0$.

Let

$$s_0 = \frac{n-d}{n},\ s_1 = \frac{n-d}{d},\ s_2 = n-d. \tag{2.3}$$

Theorem 2.1. *Let $v_1, v_2 \in L^\infty_\sigma(\mathbb{R}^d)$ for some $\sigma > d$, $v_1 - v_2 \in W^{n,1}(\mathbb{R}^d)$ for some $n > d$, supp $(v_1 - v_2) \subset D$, where D is an open bounded domain in $\mathbb{R}^d$, $d \ge 2$. Let $\|v_j\|_\sigma \le N_1$, $\|v_1 - v_2\|_{n,1} \le N_2$, where $\|\cdot\|_\sigma$, $\|\cdot\|_{n,1}$ are defined in (1.3), (2.1). Let f_1, f_2 denote the scattering amplitudes for v_1, v_2, respectively. Then:*

$$\|v_1 - v_2\|_{L^\infty(D)} \le C_1\sqrt{E}\|f_1 - f_2\|_{C(\Gamma_E^{\tau(E)})} + C_2(\sqrt{E})^{-s_1}, \tag{2.4}$$

$$\|v_1 - v_2\|_{L^\infty(D)} \le \tilde{C}_1\|f_1 - f_2\|_{C(\Gamma_E^{\tau(E)}),n_0} + C_2(\sqrt{E})^{-s_1}, d < n_0 \le n, \tag{2.5}$$

where $\|\cdot\|_{C(\Gamma_E^\tau)}$, $\|\cdot\|_{C(\Gamma_E^\tau),n_0}$ are defined by (2.2), $\tau(E) = \varepsilon(\sqrt{E})^{(1-d)/d}$, $\varepsilon = \varepsilon(N_1, D, \sigma)$, $C_1 = C_1(N_1, D, \sigma)$, $C_2 = C_2(N_1, N_2, D, \sigma, n)$, $\tilde{C}_1 = \tilde{C}_1(N_1, D, \sigma, n_0)$, s_1 is defined in (2.3), $E \ge 1$.

In Theorem 2.1, ε, C_1, C_2, $\tilde{C}_1$ denote appropriate positive constants (independent of E). In addition, in particular, $0 < \varepsilon \le 1$.

Theorem 2.1 is proved in Section 4. There is a considerable similarity between this proof and the proof of recent stability estimates of [IN1].

Note that the old approach to inverse scattering at high energies based on formula (3.3) of Section 3 yields estimates like (2.4), (2.5) with s_0 only instead of s_1 in the error term. In addition, due to (2.3), we have that

$$s_0 \le 1 \ \text{ even for } \ n \to +\infty, \quad \text{whereas} \quad s_1 \to +\infty \ \text{ for } \ n \to +\infty.$$

In Theorem 2.1, we have that $\tau(E) \to 0$ as $E \to +\infty$. Therefore, $\Gamma_E^{\tau(E)}$ is a very small part of $\Gamma_E^{\tau_1}$ for any fixed $\tau_1 \in]0,1]$ for sufficiently high energy E. Therefore, estimates (2.4), (2.5) of Theorem 2.1 can be considered as a stability result for Problem 1.2 with incomplete data.

Let

$$\mathcal{M}_E^{\tau} = \{(k,l) \in \mathcal{M}_E : \ k-l \in \mathcal{B}_{2\tau\sqrt{E}}\}, \ E > 0, \ \tau \in]0,1]. \tag{2.6}$$

Let

$$\begin{aligned} &\|f\|_{C(\mathcal{M}_E^{\tau})} = \|f\|_{C(\mathcal{M}_E^{\tau}),0}, \\ &\|f\|_{C(\mathcal{M}_E^{\tau}),\sigma} = \sup_{(k,l)\in\mathcal{M}_E^{\tau}} (1+|k-l|)^{\sigma}|f(k,l)|, \end{aligned} \tag{2.7}$$

where $E > 0$, $0 < \tau \leq 1$, $\sigma \geq 0$.

To our knowledge, estimates (2.4), (2.5) are completely new even with $\|f_1 - f_2\|_{C(\mathcal{M}_E^{\tau(E)})}$, $\|f_1 - f_2\|_{C(\mathcal{M}_E^{\tau(E)}),n_0}$ in place of $\|f_1 - f_2\|_{C(\Gamma_E^{\tau(E)})}$, $\|f_1 - f_2\|_{C(\Gamma_E^{\tau(E)}),n_0}$ (respectively).

On the other hand, for the case of Problem 1.1 with complete data, estimates (2.4), (2.5) with $\|f_1 - f_2\|_{C(\mathcal{M}_E^{1})}$, $\|f_1 - f_2\|_{C(\mathcal{M}_E^{1}),n_0}$ in place of $\|f_1 - f_2\|_{C(\Gamma_E^{\tau(E)})}$, $\|f_1 - f_2\|_{C(\Gamma_E^{\tau(E)}),n_0}$ (respectively) are less precise than related results of [N4], [N5] with the error term estimated as $O(E^{-s_2/2})$, $E \to +\infty$, where s_2 is defined in (2.3).

In addition, for Problem 1.2 with the scattering amplitude f given on $\Gamma_E^{\tau(E)}$ only, estimates (2.4), (2.5) are rather optimal even for the case of the Born approximation (that is in the linear approximation near zero potential). We recall that, in the Born approximation,

$$f(k,l) \approx \hat{v}(k-l), \quad (k,l) \in \mathcal{M}_E, \tag{2.8}$$

where

$$\hat{v}(p) = \left(\frac{1}{2\pi}\right)^d \int\limits_{\mathbb{R}^d} e^{ipx} v(x)dx, \quad p \in \mathbb{R}^d. \tag{2.9}$$

Let

$$\begin{aligned} &\|\hat{v}\|_{C(\mathcal{B}_r)} = \|\hat{v}\|_{C(\mathcal{B}_r),0}, \\ &\|\hat{v}\|_{C(\mathcal{B}_r),\sigma} = \sup_{p\in\mathcal{B}_r} (1+|p|)^{\sigma}|\hat{v}(p)| \ r > 0, \ \sigma \geq 0. \end{aligned} \tag{2.10}$$

Born approximation analogs of (2.4), (2.5) can be written as

$$\begin{aligned} &\|v_1 - v_2\|_{L^\infty(D)} \leq c_1(d)\varepsilon^d\sqrt{E}\|\hat{v}_1 - \hat{v}_2\|_{C(\mathcal{B}_{2\varepsilon(\sqrt{E})^{1/d}})} + \\ &c_2(d,n)N_2\varepsilon^{-(n-d)}(\sqrt{E})^{-s_1}, \end{aligned} \tag{2.11}$$

$$\begin{aligned} &\|v_1 - v_2\|_{L^\infty(D)} \leq \tilde{c}_1(d,n_0)\|\hat{v}_1 - \hat{v}_2\|_{C(\mathcal{B}_{2\varepsilon(\sqrt{E})^{1/d}}),n_0} + \\ &c_2(d,n)N_2\varepsilon^{-(n-d)}(\sqrt{E})^{-s_1}, \end{aligned} \tag{2.12}$$

where s_1, n, n_0, d, N_2 are the same that in (2.3)-(2.5), $0 < \varepsilon < 1$, $E \geq 1$.

3. Some results of direct scattering

We recall that, under assumptions (1.2), the Lippmann-Schwinger integral equation (1.6) is uniquely solvable for $\psi^+(\cdot,k)\in L^\infty(\mathbb{R}^d)$ for fixed $k\in\mathbb{R}^d\backslash\{0\}$; see [BS], [F2] and references therein.

We recall that the following estimate holds:

$$\begin{aligned}&\|\Lambda^{-s}G^+(k)\Lambda^{-s}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)}=O(|k|^{-1}),\\ &\text{as}\ \ |k|\to\infty,\ k\in\mathbb{R}^d,\ \ \text{for}\ \ s>1/2,\end{aligned}\tag{3.1}$$

where $G^+(k)$ denotes the integral operator with the Schwartz kernel $G^+(x-y,k)$ of (1.6), Λ denotes the multiplication operator by the function $(1+|x|^2)^{1/2}$; see [E], [J] and references therein.

As a corollary of (1.6), (3.1), under assumptions (1.2), we have that

$$\|\Lambda^{-\sigma/2}\psi^+(\cdot,k)-\Lambda^{-\sigma/2}\psi_0^+(\cdot,k)\|_{L^2(\mathbb{R}^d)}\le a_1(d,\sigma)\|v\|_\sigma|k|^{-1}\tag{3.2}$$

for $|k|\ge\rho_1(d,\sigma)\|v\|_\sigma$, $k\in\mathbb{R}^d$, where $\psi_0^+(x,k)=e^{ikx}$.

As a corollary of (1.7), (3.2), under assumptions (1.2), we have that

$$|f(k,l)-\hat v(k-l)|\le a_2(d,\sigma)(\|v\|_\sigma)^2|k|^{-1}\tag{3.3}$$

for $k,l\in\mathbb{R}^d$, $|k|=|l|\ge\rho_1(d,\sigma)\|v\|_\sigma$, where $\hat v$ is defined by (2.9).

We recall also that, under assumptions (1.2) for $v=v_j$, $j=1,2$, the following formula holds:

$$\begin{aligned}&f_2(k,l)-f_1(k,l)=\\ &\left(\frac{1}{2\pi}\right)^d\int\limits_{\mathbb{R}^d}\psi_1^+(x,-l)(v_2(x)-v_1(x))\psi_2^+(x,k)dx,\ \ k,l\in\mathbb{R}^d,\ \ k^2=l^2>0,\end{aligned}\tag{3.4}$$

where f_j, ψ_j^+ denote f and ψ^+ for $v=v_j$, $j=1,2$; see [S2].

In addition, in the proof of Theorem 2.1 we use, in particular, the following lemma:

Lemma 3.1. *Let $v=v_j$ satisfy (1.2), $\|v_j\|_\sigma\le N$, where $j=1,2$. Let supp$\,(v_1-v_2)\subset D$, where D is an open bounded domain in $\mathbb{R}^d$. Then the following estimate holds:*

$$\begin{aligned}&|(f_2(k,l)-f_1(k,l))-(\hat v_2(k-l)-\hat v_1(k-l))|\le\\ &a_3(D,\sigma)N\|v_2-v_1\|_{L^\infty(D)}|k|^{-1}\end{aligned}\tag{3.5}$$

for $k,l\in\mathbb{R}^d$, $|k|=|l|\ge\rho_1(d,\sigma)N$.

Lemma 3.1 follows from formula (3.4), estimate (3.2) and the property that $\inf\limits_{x\subset D}(1+|x|^2)^{-\sigma/4}>0$.

4. Proof of Theorem 2.1

We have that

$$\|v_2 - v_1\|_{L^\infty(D)} \le \sup_{x\in D} \Big|\int\limits_{\mathbb{R}^d} e^{-ipx}(\hat v_2(p) - \hat v_1(p))dp\Big| \le \\ I_1(\kappa) + I_2(\kappa) \ \ \text{for any} \ \ \kappa > 0, \tag{4.1}$$

where

$$I_1(\kappa) = \int\limits_{|p|\le\kappa} |\hat v_2(p) - \hat v_1(p)|dp, \\ I_2(\kappa) = \int\limits_{|p|\ge\kappa} |\hat v_2(p) - \hat v_1(p)|dp. \tag{4.2}$$

Due to Lemma 3.1, we have that

$$|\hat v_2(p) - \hat v_1(p)| \le |f_2(k_E(p), l_E(p)) - f_1(k_E(p), l_E(p))| + \\ a_3(D,\sigma)N_1\|v_2 - v_1\|_{L^\infty(D)}(\sqrt{E})^{-1} \tag{4.3}$$

for $p \in \mathcal{B}_{2\sqrt{E}}$, $\sqrt{E} \ge \rho_1(d,\sigma)N_1$, where

$$k_E(p) = \frac{p}{2} + \eta_E(p), \quad l_E(p) = -\frac{p}{2} + \eta_E(p), \tag{4.4}$$

where η_E is the function of (1.8), (1.10).

Using (4.2), (4.3), (2.2), we obtain that

$$I_1(2\tau\sqrt{E}) \le |\mathcal{B}_1|(2\tau\sqrt{E})^d\big(\|f_2 - f_1\|_{C(\Gamma_E^\tau)} + \frac{a_3(D,\sigma)N_1\|v_2 - v_1\|_{L^\infty(D)}}{\sqrt{E}}\big), \tag{4.5}$$

$$I_1(2\tau\sqrt{E}) \le \frac{|\mathbb{S}^{d-1}|}{n_0 - d}\|f_2 - f_1\|_{C(\Gamma_E^\tau),n_0} + |\mathcal{B}_1|a_3(D,\sigma)N_1\frac{(2\tau\sqrt{E})^d}{\sqrt{E}}\|v_2 - v_1\|_{L^\infty(D)} \tag{4.6}$$

for $\sqrt{E} \ge \rho_1(d,\sigma)N_1$, $\tau \in]0,1]$, where $|\mathcal{B}_1|$ and $|\mathbb{S}^{d-1}|$ denote standard Euclidean volumes of $\mathcal{B}_1$ and $\mathbb{S}^{d-1}$ (respectively), n_0 is the number of (2.5).

The assumptions that $v_1 - v_2 \in W^{n,1}(\mathbb{R}^d)$, $\|v_1 - v_2\|_{n,1} \le N_2$ for some $n > d$, imply that

$$|\hat v_2(p) - \hat v_1(p)| \le a_4(n,d)N_2(1 + |p|)^{-n}, \quad p \in \mathbb{R}^d. \tag{4.7}$$

Using (4.2), (4.7) we obtain that

$$I_2(2\tau\sqrt{E}) \le \frac{|\mathbb{S}^{d-1}|a_4(n,d)N_2}{n - d}\frac{1}{(2\tau\sqrt{E})^{n-d}}, \tag{4.8}$$

$\sqrt{E} > 0$, $\tau \in]0,1]$.

Let

$$\begin{aligned}
&\tau(E) = \varepsilon(\sqrt{E})^{(1-d)/d},\\
&\varepsilon = \min\left(\frac{1}{2}\left(\frac{1}{2|\mathcal{B}_1|a_3(D,\sigma)N_1}\right)^{1/d}, 1\right).
\end{aligned} \tag{4.9}$$

Due to (4.9), we have, in particular, that

$$\begin{aligned}
&|\mathcal{B}_1|a_3(D,\sigma)N_1(2\tau(E)\sqrt{E})^d(\sqrt{E})^{-1} \le \frac{1}{2},\\
&\tau(E) \le 1, \quad E \ge 1.
\end{aligned} \tag{4.10}$$

Using (4.1), (4.5), (4.6), (4.8), (4.10), we obtain that

$$\begin{aligned}
&\|v_2 - v_1\|_{L^\infty(D)} \le \frac{\sqrt{E}}{a_3(D,\sigma)N_1}\|f_2 - f_1\|_{C(\Gamma_E^{\tau(E)})} +\\
&\frac{1}{2}\|v_2 - v_1\|_{L^\infty(D)} + \frac{|\mathbb{S}^{d-1}|a_4(n,d)N_2}{(n-d)(2\varepsilon)^{n-d}}\frac{1}{(\sqrt{E})^{(n-d)/d}},
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
&\|v_2 - v_1\|_{L^\infty(D)} \le \frac{|\mathbb{S}^{d-1}|}{n_0 - d}\|f_2 - f_1\|_{C(\Gamma_E^{\tau(E)}),n_0} +\\
&\frac{1}{2}\|v_2 - v_1\|_{L^\infty(D)} + \frac{|\mathbb{S}^{d-1}|a_4(n,d)N_2}{(n-d)(2\varepsilon)^{n-d}}\frac{1}{(\sqrt{E})^{(n-d)/d}}
\end{aligned} \tag{4.12}$$

for $\sqrt{E} \ge \max(\rho_1(d,\sigma)N_1, 1)$.

Estimates (2.4), (2.5) with

$$C_1 = \frac{2}{a_3(D,\sigma)N_1}, \quad C_2 = \frac{2|\mathbb{S}^{d-1}|a_4(n,d)N_2}{(n-d)(2\varepsilon)^{n-d}}, \quad \tilde{C}_1 = \frac{2|\mathbb{S}^{d-1}|}{n_0 - d}, \tag{4.13}$$

for $\sqrt{E} \ge \max(\rho_1(d,\sigma)N_1, 1)$, follow from (4.11), (4.12).

Using also that $\|v_2 - v_1\|_{L^\infty(D)} \le 2N_1$ we obtain estimates (2.4), (2.5) with

$$\begin{aligned}
&C_1 = \frac{2}{a_3(D,\sigma)N_1}, \quad \tilde{C}_1 = \frac{2|\mathbb{S}^{d-1}|}{n_0 - d},\\
&C_2 = \max\left(\frac{2|\mathbb{S}^{d-1}|a_4(n,d)N_2}{(n-d)(2\varepsilon)^{n-d}},\ 2N_1(\max(\rho_1(d,\sigma)N_1, 1))^{(n-d)/d}\right)
\end{aligned} \tag{4.14}$$

for $E \ge 1$.

This completes the proof of Theorem 2.1.

Acknowledgements

This work was partially supported by TFP No 14.A18.21.0866 of Ministry of Education and Science of Russian Federation.

References

[ABR] N.V. Alexeenko, V.A. Burov, O.D. Rumyantseva, Solution of the three-dimensional acoustical inverse scattering problem. The modified Novikov algorithm, Acoust. J. 54(3), 2008, 469-482 (in Russian), English transl.: Acoust. Phys. 54(3), 2008, 407-419.

[BK] L. Beilina, M.V. Klibanov, Approximate global convergence and adaptivity for coefficient inverse problems, Springer (New York), 2012.

[BS] F.A. Berezin, M.A. Shubin, The Schrödinger Equation, Vol. 66 of Mathematics and Its Applications, Kluwer Academic, Dordrecht, 1991.

[B] A.L. Buckhgeim, Recovering a potential from Cauchy data in the two-dimensional case, J. Inverse Ill-Posed Probl. 16(1), 2008, 19-33.

[BAR] V.A. Burov, N.V. Alekseenko, O.D. Rumyantseva, Multifrequency generalization of the Novikov algorithm for the two-dimensional inverse scattering problem, Acoustical Physics 55(6), 2009, 843-856.

[E] G. Eskin, Lectures on Linear Partial Differential Equations, Graduate Studies in Mathematics, Vol.123, American Mathematical Society, 2011.

[ER1] G. Eskin, J. Ralston, Inverse backscattering problem in three dimensions, Commun. Math. Phys. 124, 1989, 169-215.

[ER2] G. Eskin, J. Ralston, Inverse scattering problem for the Schrödinger equation with magnetic potential at a fixed energy, Commun. Math. Phys. 173, 1995, 199-224.

[F1] L.D. Faddeev, Uniqueness of the solution of the inverse scattering problem, Vest. Leningrad Univ. 7, 1956, 126-130 [in Russian].

[F2] L.D. Faddeev, The inverse problem in the quantum theory of scattering.II, Current problems in mathematics, Vol. 3, 1974, pp. 93-180, 259. Akad. Nauk SSSR Vsesojuz. Inst. Naucn. i Tehn. Informacii, Moscow(in Russian); English Transl.: J.Sov. Math. 5, 1976, 334-396.

[FM] L.D. Faddeev, S.P. Merkuriev, Quantum Scattering Theory for Multi-particle Systems", Nauka, Moscow, 1985 [in Russian].

[G] P.G. Grinevich, The scattering transform for the two-dimensional Schrödinger operator with a potential that decreases at infinity at fixed nonzero energy, Uspekhi Mat. Nauk 55:6(336),2000, 3-70 (Russian); English translation: Russian Math. Surveys 55:6, 2000, 1015-1083.

[HH] P. Hähner, T. Hohage, New stability estimates for the inverse acoustic inhomogeneous medium problem and applications, SIAM J. Math. Anal., 33(3), 2001, 670-685.

[HN] G.M. Henkin, R.G. Novikov, The $\bar{\partial}$-equation in the multidimensional inverse scattering problem, Uspekhi Mat. Nauk 42(3), 1987, 93-152 (in Russian); English Transl.: Russ. Math. Surv. 42(3), 1987, 109-180.

[I] M.I. Isaev, Exponential instability in the inverse scattering problem on the energy interval, Func. Analiz i ego Pril.(to appear), arXiv:1012.5526.

[IN1] M.I. Isaev, R.G. Novikov, Energy and regularity dependent stability estimates for the Gelfănd inverse problem in multidimensions, J. Inverse and III-posed Probl., 20(3), 2012, 313-325.

[IN2] M.I. Isaev, R.G. Novikov New global stability estimates for monochromatic inverse acoustic scattering, e-print arXiv:1210.3821.

[J] A. Jensen, High energy resolvent estimates for generalized many-body Schrödinger operators, Publ. RIMS Kyoto Univ. 25, 1989, 155-167.

[K] M.V. Klibanov, Carleman estimates for global uniqueness, stability and numerical methods for coefficient inverse problems, J. Inverse and Ill-posed Probl., (to appear).

[LRS] M.M. Lavrentev, V.G. Romanov, S.P. Shishatskii, Ill-posed problems of mathematical physics and analysis, Translated from the Russian by J.R. Schulenberger. Translation edited by Levi J. Leifman. Translations of Mathematical Monographs, 64. American Mathematical Society, Providence, RI, 1986.

[N1] R.G. Novikov, Multidimensional inverse spectral problem for the equation $-\Delta\psi + (v(x) - Eu(x))\psi = 0$, Funkt. Anal. Prilozhen. 22(4), 1988, 11-22 (in Russian); Engl. Transl. Funct. Anal. Appl. 22, 1988, 263-272.

[N2] R.G. Novikov, The inverse scattering problem at fixed energy level for the two-dimensional Schrödinger operator, J. Funct. Anal., 103, 1992, 409-463.

[N3] R.G. Novikov, The inverse scattering problem at fixed energy for Schrödinger equation with an exponentially decreasing potential, Comm. Math.
Phys., 161, 1994, 569-595.

[N4] R.G. Novikov, Rapidly converging approximation in inverse quantum scattering in dimension 2. Physics Letters A 238, 1998, 73-78.

[N5] R.G. Novikov, The $\bar\partial$-approach to approximate inverse scattering at fixed energy in three dimensions, IMRP Int. Math. Res. Pap. 2005, no. 6, 287-349.

[N6] R.G. Novikov, On non-overdetermined inverse scattering at zero energy in three dimensions, Ann. Scuola Norm. Sup. Pisa CL. Sci. 5, 2006, 279-328.

[S1] P. Stefanov, Stability of the inverse problem in potential scattering at fixed energy, Annales de l'Institut Fourier, tome 40(4), 1990, 867-884.

[S2] P. Stefanov, A uniqueness result for the inverse back-scattering problem, Inverse Problems 6, 1990, 1055-1064.

[VW] A.Vasy, X.-P. Wang, Inverse scattering with fixed energy for dilation-analytic potentials, Inverse Problems, 20, 2004, 1349-1354.

[W] R. Weder, Global uniqueness at fixed energy in multidimensional inverse scattering theory, Inverse Problems 7, 1991, 927-938.

[WY] R. Weder, D. Yafaev, On inverse scattering at a fixed energy for potentials with a regular behaviour at infinity, Inverse Problems, 21, 2005, 1937-1952.